

TOWARDS TRIGONOMETRIC DEFORMATION OF  $\widehat{\mathfrak{sl}}_2$  COSET VOA

B. FEIGIN, M. JIMBO, AND E. MUKHIN

ABSTRACT. We discuss the quantization of the  $\widehat{\mathfrak{sl}}_2$  coset vertex operator algebra  $WD(2,1;\alpha)$  using the bosonization technique. We show that after quantization there exist three families of commuting integrals of motion coming from three copies of the quantum toroidal algebra associated to  $\mathfrak{gl}_2$ .

## 1. INTRODUCTION

In this paper we study the trigonometric deformation of the coset algebra in the case of  $\mathfrak{sl}_2$ .

Let  $\mathfrak{g}$  be a finite-dimensional semisimple Lie algebra. Fix three levels  $k_1, k_2, k_3$  such that

$$(1.1) \quad k_1 + k_2 + k_3 = -2h^\vee,$$

where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$ . By definition, the vacuum representation of the coset algebra  $C(\mathfrak{g}; k_1, k_2, k_3)$  is the space of semi-infinite cohomology  $H^{\infty/2}(\widehat{\mathfrak{g}}, \mathfrak{g}, V_{k_1} \otimes V_{k_2} \otimes V_{k_3})$ , where  $V_{k_i}$  are the vacuum representations of  $\widehat{\mathfrak{g}}$  of level  $k_i$ . In this paper we consider only generic  $k_1, k_2, k_3$  satisfying (1.1), when the semi-infinite cohomology is non-trivial only in one dimension.

The above definition is symmetric in  $k_1, k_2, k_3$ . A more standard definition of the coset algebra is non-symmetric. Namely, one defines  $C(\mathfrak{g}; k_1, k_2, k_3)$  to be the space of invariants  $(V_{k_1} \otimes V_{k_2})^{\mathfrak{g} \otimes \mathbb{C}[t]}$ . Informally, the coset algebra is the commutant of diagonal  $\widehat{\mathfrak{g}}$  inside  $U\widehat{\mathfrak{g}}_{k_1} \otimes U\widehat{\mathfrak{g}}_{k_2}$ , where  $U\widehat{\mathfrak{g}}_{k_i}$  is the universal enveloping algebra of  $\widehat{\mathfrak{g}}$  on level  $k_i$ .

There is yet another definition of the coset algebra  $C(\mathfrak{g}; k_1, k_2, k_3)$  as a subalgebra of  $V_{k_1}$ . For example, in the limit  $k_2 \rightarrow \infty$ , the coset algebra  $C(\mathfrak{g}; k_1, \infty, \infty)$  is identified with the subalgebra of invariants  $(V_{k_1})^{\mathfrak{g}}$ . For general  $k_1, k_2, k_3$  one needs to use the machinery of the screenings. This is the construction used in this paper, so let us describe the main points.

Quite generally, given a vertex operator algebra (VOA)  $V$  and a set of fields  $s_i(z)$ ,  $i = 1, \dots, n$ , called screening currents, one defines screenings  $S_i = \int s_i(z) dz$  and the algebra  $V(S_1, \dots, S_n)$  consisting of all local fields which commute with all screenings:  $V(S_1, \dots, S_n) = \{a(z) \mid [a(z), S_i] = 0, i = 1, \dots, n\}$ . The algebra  $V(S_1, \dots, S_n)$  is closed under the operator product, and if it contains a Virasoro current, then it is a VOA.

Let us assume that  $\mathfrak{g}$  is simply laced. Let  $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$  be the Heisenberg algebra generated by Cartan currents and let  $H$  be a lattice VOA of  $\widehat{\mathfrak{h}}$ . Let  $s_i(z) = \exp(c_+ \alpha_i(z))$ ,  $i = 1, \dots, n$ ,  $n = \text{rk}(\mathfrak{g})$ ,  $c_+ \in \mathbb{C}^\times$ , be the vertex operators corresponding to rescaled simple roots of  $\mathfrak{g}$ . Then  $H(S_1, \dots, S_n) = W(\mathfrak{g})$  is the W-algebra which can be alternatively obtained by the quantum Drinfeld-Sokolov reduction of  $U\widehat{\mathfrak{g}}$  with respect to a maximal nilpotent subalgebra  $\widehat{\mathfrak{n}}$ , see [FF1]. Consider the dual screening currents  $s_i^\vee(z) = \exp(-c_- \alpha_i(z))$ , and dual screenings  $S_i^\vee = \int s_i^\vee(z) dz$ , where  $c_- \in \mathbb{C}^\times$  is another parameter.

Consider the algebra  $\tilde{V} = V_{k_1} \otimes H$  with screening currents  $\tilde{s}_i(z) = e_i(z)s_i(z)$ ,  $\tilde{s}_i^\vee(z) = f_i(z)s_i^\vee(z)$  where  $e_i(z)$ ,  $f_i(z)$  are the  $\widehat{\mathfrak{g}}$  currents corresponding to the positive and negative simple root vectors. Let  $\tilde{S}_i = \int \tilde{s}_i(z) dz$ ,  $\tilde{S}_i^\vee = \int \tilde{s}_i^\vee(z) dz$  be the corresponding screenings. Then  $[\tilde{S}_i, \tilde{S}_j^\vee] = 0$  and  $\tilde{V}(\tilde{S}_1, \dots, \tilde{S}_n, \tilde{S}_1^\vee, \dots, \tilde{S}_n^\vee)$  is the coset algebra  $C(\mathfrak{g}; k_1, k_2, k_3)$  together with the diagonal copy of the  $\widehat{\mathfrak{h}}$ , where  $c_+, c_-$  and  $k_2, k_3$  are related by  $(c_+)^2(k_2 + h^\vee) = 1$ ,  $(c_-)^2(k_3 + h^\vee) = -1$ .

This is the author's manuscript of the article published in final edited form as:

In particular, using the Wakimoto realization of  $V_{k_1}$ , see [W], [FF], we obtain a free field realization of the coset algebra.

A similar, more involved, construction exists in the non-simply laced and in the supersymmetric cases.

On mathematically rigorous level, the correspondence between three different definitions of the coset algebra is not fully established. For the latest results, see [ACL].

There exists a deep connection between VOAs and quantum groups. In particular, there exists a relationship between the quasi-tensor category of representations of a VOA and the tensor category of representations of a quantum group. The screening construction is a reflection of this phenomenon. In general, the screenings which define the VOA generate (in some appropriate sense) the nilpotent part of the quantum group. In our situation, the coset algebra is a reduction of  $V_{k_1} \otimes V_{k_2} \otimes V_{k_3}$ , therefore we expect to have screenings which generate the nilpotent parts of three commuting quantum groups  $U_{q^{(j)}} \mathfrak{g}$  with  $q^{(j)} = \exp(2\pi i/(k_j + h^\vee))$ ,  $j = 1, 2, 3$ .

This is indeed so in the case of  $\mathfrak{sl}_2$ , see [FS] and Section 2. The algebra  $U(\widehat{\mathfrak{sl}}_2)_{k_1}$  has the Wakimoto realization through three free fields. Then there are three screening currents:  $\tilde{s}_1(z)$ ,  $\tilde{s}_1^\vee(z)$ , and the Wakimoto screening current. These screening currents have the form  $\theta_i(z) \exp(\tilde{\theta}_i(z))$ , where  $\theta_i(z)$ ,  $\tilde{\theta}_i(z)$  are free fields, and the corresponding screenings commute. We call these screenings “bosonic”.

Given an embeddings of VOAs  $W_1 \subset W_2$ , one can ask whether it is possible to find all screenings: that is, to find all fields  $s(z)$  in  $W_2$  such that  $\int s(z)dz$  commutes with  $W_1$ . Unfortunately, we know no results in this direction. However, in the present situation, we observe a dual “fermionic picture”. Namely, there are three more screening currents which anticommute, and the corresponding screenings form the nilpotent part of the quantum Lie superalgebra of type  $D(2, 1; \alpha)$ . The fermionic currents are simple exponentials. In particular, we expect that the  $\mathfrak{sl}_2$  coset algebra can be obtained by the quantum Drinfeld-Sokolov reduction from  $\hat{D}(2, 1; \alpha)$ .

The next important ingredient is the description of integrable systems —the commutative subalgebras of vertex operator algebras. Such subalgebras are also defined as commutants of a set of screenings. In general the set of local fields commuting with all screenings may be trivial, but under certain circumstances there exists an infinite series of descendants  $a_j(z)$  of the vacuum such that  $I_j = \int a_j(z)dz$  commute with all screenings. The operators  $I_j$  are called local integrals of motion. In all known cases, the set of screenings leading to an infinite set of local integrals of motion corresponds to an affine quantum group.

There exist also non-local integrals of motion given by suitable integrals of products of screening currents, see [BLZ]. Conjecturally, the cycles of integration defining the non-local integrals of motion correspond to elements of the center of the affine quantum group at the critical level.

In the case of  $\mathfrak{sl}_2$  this is achieved by adjoining one more fermionic screening, or three more bosonic screenings. Four fermionic screenings correspond to nilpotent part of the quantum affine superalgebra  $\hat{D}(2, 1; \alpha)$ , and three pairs of bosonic screenings correspond to the nilpotent parts of three commuting copies of the quantum affine algebra  $\widehat{\mathfrak{sl}}_2$ . For the latest developments about the integrals of motion, see [BL], [BKL].

In our previous papers [FJMM], [FJMM1], [FJM] we have studied the trigonometric deformation of W algebras  $W\mathfrak{sl}_n$ . The advantage of deformation is that the non-local integrals of motion can be obtained from the R-matrix formalism using quantum toroidal algebras. In particular, the non-local integrals of motion can be described by appropriate explicit integrals, we can prove they commute and, moreover, we can study their spectrum by the machinery of Bethe ansatz. The aim of the present paper is to discuss the trigonometric deformation of the picture mentioned above for the coset W algebra.

We use four free fields and a three dimensional lattice to construct the following objects.

- Four fermionic screening currents which are simple exponentials. We call the corresponding screenings  $\sigma_i$ ,  $i = 0, 1, 2, 3$ .

- Three pairs of bosonic screening currents which are sums of two exponential terms. The corresponding screenings  $\rho_i, \tau_i$ ,  $i = 1, 2, 3$ , anticommute with fermionic screenings and, in addition, satisfy  $[\rho_i, \rho_j] = [\tau_i, \tau_j] = 0$  for all  $i, j$ , and  $[\rho_i, \tau_j] = 0$  for  $i \neq j$ .
- Three quantum affine algebras  $U_{q^{(i)}} \widehat{\mathfrak{sl}}_2$  on level  $q^{k_i}$ ,  $i = 1, 2, 3$ , in Wakimoto realization. Algebra  $U_{q^{(i)}} \widehat{\mathfrak{sl}}_2$  commutes with one bosonic screening  $\rho_i$  and two fermionic screenings  $\sigma_j$ ,  $j \neq i, 0$ . The screening currents corresponding to  $\rho_j$ ,  $j \neq i$  are obtained by dressing the  $E(z)$  and  $F(z)$  currents of  $U_q^{(i)} \widehat{\mathfrak{sl}}_2$  by an appropriate free field.
- Three quantum toroidal  $\mathfrak{gl}_2$  algebras  $\mathcal{E}_{2,i}(q^{-k_i-2}, q^2, q^{k_i})$  on level  $q^{k_i}$ ,  $i = 1, 2, 3$ . The screening currents corresponding to  $\rho_i$  and  $\tau_i$  are obtained by dressing the currents  $F_1(z)$  and  $F_0(z)$  of  $\mathcal{E}_{2,i}$ .
- Three sets of integrals of motion  $\mathbb{G}_{i,M}^\nu$ ,  $i = 1, 2, 3$ ,  $\nu = 0, 1$ ,  $M \in \mathbb{Z}_{\geq 1}$  coming from  $\mathcal{E}_{2,i}$ . All integrals of motion commute among themselves.

We call the commutativity of the three sets of integrals the triality. The triality is very different from standard duality described in [FJM2]. In some sense, the triality corresponds to the commutativity of three sets of integrals of motion coming from the same (unknown) algebra. We expect the existence of yet another set of integrals of motion coming from quantum toroidal  $D(2, 1; \alpha)$  which should be in the standard duality with the three sets of integrals described in this paper.

There are several open questions: (1) to give an explicit description of the deformed coset algebra, that is, to construct local currents which commute with three fermionic screenings; (2) to find non-local integrals of motion obtained from quantum toroidal  $D(2, 1; \alpha)$ ; and (3) to obtain local integrals of motion. We hope to address these issues in some other occasion.

The paper is constructed as follows. In Section 2 we recall some formulas from [FS] in the non-deformed picture. In Section 3 we give the quantization of these formulas. We construct three actions of quantum affine  $\mathfrak{sl}_2$  and extend them to the action of quantum toroidal  $\mathfrak{gl}_2$  in Section 4. That allows us to show that the three integrable systems commute in Section 5. In Section 6 we discuss the connection to quantum toroidal  $D(2, 1; \alpha)$ . The Appendices contain some technical formulas and computations.

## 2. W ALGEBRA $WD(2, 1; \alpha)$

The W algebra  $WD(2, 1; \alpha)$ , termed also “corner-brane  $W$  algebra” [LZ], has many intriguing features. From conformal field theory (CFT) point of view, it can be viewed as a coset theory  $(\mathfrak{sl}_2)_{k_1} \times (\mathfrak{sl}_2)_{k_2} / (\mathfrak{sl}_2)_{k_1+k_2}$  with generic parameters  $k_1, k_2$ . In [FS] it was identified with a quantum Hamiltonian reduction of the exceptional affine Lie superalgebra  $\widehat{D}(2, 1; \alpha)$ . Algebra  $WD(2, 1; \alpha)$  governs the CFT limit of an integrable quantum field theory introduced in [Fa]. In a remarkable work [BL] the ODE/IM correspondence for this CFT model was found to be related to generalized hypergeometric Opers, and was further extended to include Fateev’s massive integrable field theory.

The aim of this section is to summarize basic aspects of  $WD(2, 1; \alpha)$  following [FS] and motivate the construction in the main text.

Algebra  $WD(2, 1; \alpha)$  is realized in the Fock space of three bosonic free fields  $\varphi_i(z) = Q_i + \varphi_{i,0} \log z - \sum_{n \neq 0} \frac{\varphi_{i,n}}{n} z^{-n}$  ( $i = 1, 2, 3$ ) satisfying  $[\varphi_{i,0}, Q_j] = \delta_{i,j}$ ,  $[\varphi_{i,m}, \varphi_{j,n}] = \delta_{i,j} n \delta_{m+n,0}$ . The total Fock space decomposes into subspaces

$$\bar{\mathcal{H}} = \bigoplus_{\beta} \bar{\mathcal{H}}_{\beta}, \quad \bar{\mathcal{H}}_{\beta} = \mathbb{C}[\varphi_{i,-n} \mid n > 0, i = 1, 2, 3] \otimes \mathbb{C}e^{\beta},$$

where the “momentum”  $\beta$  runs over the three dimensional space  $\oplus_{i=1}^3 \mathbb{C}Q_i$ . We shall refer to  $\bar{\mathcal{H}}_{\beta}$ ’s as “sectors”.

Fix complex numbers  $k_i$  ( $i = 1, 2, 3$ ) satisfying  $k_1 + k_2 + k_3 = -4$ . Choose vectors  $\mathbf{a}_i \in \mathbb{C}^3$  ( $i = 1, 2, 3$ ) with scalar product

$$\mathbf{a}_i \cdot \mathbf{a}_i = 1, \quad \mathbf{a}_1 \cdot \mathbf{a}_2 = k_3 + 1 \quad \& \text{cycl..}$$

Here and after we shall use the symbol “& cycl.” to imply all equations obtained by cyclically permuting the indices  $(1, 2, 3)$ . Using the vector notation  $\boldsymbol{\varphi}(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))$ , we introduce fermionic screening operators  $\sigma_i$  ( $i = 1, 2, 3$ ) by

$$(2.1) \quad \sigma_i = \int : e^{\mathbf{a}_i \cdot \boldsymbol{\varphi}(z)} : dz.$$

Next let  $\mathbf{r}_1 = -\frac{1}{k_1+2}(\mathbf{a}_2 + \mathbf{a}_3)$  & cycl.. These vectors are mutually orthogonal. We introduce bosonic screening operators  $\rho_i$  ( $i = 1, 2, 3$ ) by

$$(2.2) \quad \rho_1 = \int : (\mathbf{a}_3 \cdot \partial \boldsymbol{\varphi}(z)) e^{\mathbf{r}_1 \cdot \boldsymbol{\varphi}(z)} : dz \quad \& \text{cycl..}$$

Now let  $\alpha = k_1/(k_3 + 2)$ . The W algebra  $\mathcal{WD}(2, 1; \alpha)$  is a vertex subalgebra in the vacuum sector  $\bar{\mathcal{H}}_0$ . It is defined either as the intersection of the kernels of  $\sigma_i$ , or that of the kernels of  $\rho_i$  [FS],

$$\mathcal{WD}(2, 1; \alpha) = \bigcap_{i=1}^3 \text{Ker} \left( \sigma_i|_{\bar{\mathcal{H}}_0} \right) = \bigcap_{i=1}^3 \text{Ker} \left( \rho_i|_{\bar{\mathcal{H}}_0} \right) \subset \bar{\mathcal{H}}_0.$$

As is well known, W algebras associated with simple Lie algebras admit two descriptions in terms of quantum groups which are Langlands dual to each other. In the present case the relevant quantum groups look rather different. The fermionic screening operators  $\{\sigma_i\}_{i=1,2,3}$  generate the nilpotent part  $U_{\mathbf{q}}^+ D(2, 1; \alpha)$  of the quantum group associated with the exceptional Lie superalgebra  $D(2, 1; \alpha)$  (whence the notation  $\mathcal{WD}(2, 1; \alpha)$ ). In turn, the bosonic screening operators  $\{\rho_i\}_{i=1,2,3}$  generate  $U_{\mathbf{q}_1}^+ \mathfrak{sl}_2 \otimes U_{\mathbf{q}_2}^+ \mathfrak{sl}_2 \otimes U_{\mathbf{q}_3}^+ \mathfrak{sl}_2$  where  $\mathbf{q}_i = \exp(2\pi i/(k_i + 2))$ .

It is easy to see that  $\mathcal{WD}(2, 1; \alpha)$  contains the Virasoro current

$$T_2(z) = \frac{1}{2} \sum_{i=1}^3 \frac{k_i + 2}{2} : \partial r_i(z)^2 : + \frac{1}{2} \sum_{i=1}^3 \partial^2 r_i(z), \quad r_i(z) = \mathbf{r}_i \cdot \boldsymbol{\varphi}(z),$$

with central charge  $c = 3 - \sum_{i=1}^3 \frac{6}{k_i+2}$ . As vertex algebra,  $\mathcal{WD}(2, 1; \alpha)$  is conjecturally generated by  $T_2(z)$  and a current  $T_4(z)$  of spin 4. An explicit expression for  $T_4(z)$  can be found in [FS], [LZ].

In order to discuss integrals of motion, we need additional screening operators. Let  $\sigma_0$  be given by the same formula (2.1) with  $\mathbf{a}_0 = -\sum_{i=1}^3 \mathbf{a}_i$ . Let further

$$\tau_i = \int : (\mathbf{a}_i \cdot \partial \boldsymbol{\varphi}(z)) e^{-\mathbf{r}_i \cdot \boldsymbol{\varphi}(z)} : dz$$

for  $i = 1, 2, 3$ . Then  $\{\sigma_i\}_{i=0,1,2,3}$  generate  $U_{\mathbf{q}}^+ \hat{D}(2, 1; \alpha)$ , and  $\{\rho_i, \tau_i\}_{i=1,2,3}$  generate  $U_{\mathbf{q}_1}^+ \hat{\mathfrak{sl}}_2 \otimes U_{\mathbf{q}_2}^+ \hat{\mathfrak{sl}}_2 \otimes U_{\mathbf{q}_3}^+ \hat{\mathfrak{sl}}_2$ . One expects that their joint kernels coincide,

$$\bigcap_{i=0}^3 \text{Ker} \left( \sigma_i|_{\bar{\mathcal{H}}_0} \right) = \bigcap_{i=1}^3 \left( \text{Ker} \left( \rho_i|_{\bar{\mathcal{H}}_0} \right) \cap \text{Ker} \left( \tau_i|_{\bar{\mathcal{H}}_0} \right) \right),$$

giving a commutative algebra generated by integrals of local densities

$$\mathbb{I}_n = \int P_{n+1}(z) dz \quad (n = 1, 3, 5, \dots).$$

They are called “local” integrals of motion. One expects also the existence of an infinite family of “non-local” integrals of motion given by multiple integrals of products of screening currents. The bosonic ones

associated with  $U_{\mathfrak{q}_1}^+ \widehat{\mathfrak{sl}}_2 \otimes U_{\mathfrak{q}_2}^+ \widehat{\mathfrak{sl}}_2 \otimes U_{\mathfrak{q}_3}^+ \widehat{\mathfrak{sl}}_2$  are discussed in [BL]. Little is known about non-local integrals of motion associated with  $U_{\mathfrak{q}}^+ \widehat{D}(2, 1; \alpha)$ .

### 3. DEFORMED SCREENING CURRENTS

In this section, we  $q$ -deform the screening currents considered in the previous section. We fix  $q \in \mathbb{C}^\times$  satisfying  $|q| < 1$ , and use the standard symbol  $[x] = (q^x - q^{-x})/(q - q^{-1})$ .

Unlike the conformal case, we use here *four* free fields

$$a_i(z) = Q_{a_i} + a_{i,0} \log z - \sum_{n \neq 0} \frac{a_{i,n}}{n} z^{-n}, \quad i = 0, 1, 2, 3.$$

We choose  $\{a_{i,n} \mid n \neq 0, i = 0, 1, 2, 3\}$  to be an independent set of oscillators. Concerning the zero modes, however, we assume linear relations

$$a_{0,0} + a_{1,0} + a_{2,0} + a_{3,0} = 0, \quad Q_{a_0} + Q_{a_1} + Q_{a_2} + Q_{a_3} = 0.$$

Hence the entire Fock space decomposes into sectors

$$\mathcal{H} = \bigoplus_{\beta} \mathcal{H}_{\beta}, \quad \mathcal{H}_{\beta} = \mathbb{C}[a_{i,-n} \mid n > 0, i = 0, 1, 2, 3] \otimes \mathbb{C}e^{\beta},$$

with the momentum  $\beta$  running over the same three dimensional space  $\oplus_{i=1}^3 \mathbb{C}Q_{a_i}$  as in the CFT case.

We set the following commutation relations among the Fourier modes,

$$(3.1) \quad [a_{i,n}, a_{i,-n}] = n, \quad [a_{i,0}, Q_{a_i}] = 1,$$

$$(3.2) \quad [a_{0,n}, a_{i,-n}] = n \frac{[(k_i + 1)n]}{[n]}, \quad [a_{0,0}, Q_{a_i}] = k_i + 1,$$

$$(3.3) \quad [a_{1,n}, a_{2,-n}] = n \frac{[(k_3 + 1)n]}{[n]}, \quad [a_{1,0}, Q_{a_2}] = k_3 + 1 \quad \& \text{ cycl..}$$

All other commutators are zero.

Fermionic screening currents in the deformed case are still pure exponentials,

$$(3.4) \quad \sigma_i(z) =: e^{a_i(z)}:, \quad i = 0, 1, 2, 3.$$

Eq. (3.1) ensures that the  $\sigma_i(z)$ 's are ordinary fermions. On the other hand, deformed bosonic screening currents  $\rho_i(z)$ ,  $\tau_i(z)$  ( $i = 1, 2, 3$ ) are sums of two exponential terms

$$(3.5) \quad \rho_i(z) = \frac{1}{(q - q^{-1})z} (: e^{r_i^+(z)} : - : e^{r_i^-(z)} :),$$

$$(3.6) \quad \tau_i(z) = \frac{1}{(q - q^{-1})z} (: e^{t_i^+(z)} : - : e^{t_i^-(z)} :).$$

The exponents are given by

$$\begin{aligned} r_1^{\pm}(z) &= -a_2(k_1 + 2; z) - a_3(k_1 + 2; q^{\mp(k_1+2)} z) \quad \& \text{ cycl.}, \\ t_i^{\pm}(z) &= -a_0(k_i + 2; z) - a_i(k_i + 2; q^{\mp(k_i+2)} z), \end{aligned}$$

where

$$a_i(N; z) = \frac{1}{N}(Q_{a_i} + a_{i,0} \log z) - \sum_{n \neq 0} \frac{[n]}{[Nn]} \frac{a_{i,n}}{n} z^{-n}.$$

It is easy to see that (3.4), (3.5), (3.6) tend to the CFT counterparts in the limit  $q \rightarrow 1$ . Screening operators are defined by

$$\sigma_i = \int \sigma_i(z) dz, \quad \rho_j = \int \rho_j(z) dz, \quad \tau_j = \int \tau_j(z) dz.$$

These operators make sense on sectors  $\mathcal{H}_\beta$  with appropriate  $\beta$ . More precisely,  $\sigma_i$  is well-defined if  $[a_{i,0}, \beta] \in \mathbb{Z}$ , while  $\rho_1, \tau_1$  are well-defined if  $[a_{2,0} + a_{3,0}, \beta] \in (k_1 + 2)\mathbb{Z}$ .

**Proposition 3.1.** *The following relations hold true*

$$\begin{aligned} [\sigma_i, \rho_j]_+ &= [\sigma_i, \tau_j]_+ = 0 \quad (i = 0, 1, 2, 3, \quad j = 1, 2, 3), \\ [\rho_i, \rho_j] &= [\rho_i, \tau_j] = [\tau_i, \tau_j] = 0 \quad (i, j = 1, 2, 3, i \neq j), \end{aligned}$$

on each sector  $\mathcal{H}_\beta$  such that the left hand sides are defined.

*Proof.* We use contraction rules given in Table 1 and Table 2 in Appendix C, along with the following identities.

$$(3.7) \quad r_1^+(z) - r_1^-(z) = a_3(qz) - a_3(q^{-1}z),$$

$$(3.8) \quad r_1^+(q^{k_1+2}z) - r_1^-(q^{-k_1-2}z) = a_2(q^{-1}z) - a_2(qz),$$

$$(3.9) \quad t_1^+(z) - t_1^-(z) = a_1(qz) - a_1(q^{-1}z),$$

$$(3.10) \quad t_1^+(q^{k_1+2}z) - t_1^-(q^{-k_1-2}z) = a_0(q^{-1}z) - a_0(qz).$$

As an example let us verify the anti-commutativity between  $\sigma_2$  and  $\rho_1$ . We find from Table 1

$$\begin{aligned} (q - q^{-1})z\rho_1(z) \cdot \sigma_2(w) &= \frac{1}{q^{-k_1-1}z - w} : e^{r_1^+(z)+a_2(w)} : - \frac{1}{q^{k_1+1}z - w} : e^{r_1^-(z)+a_2(w)} :, \\ \sigma_2(w) \cdot (q - q^{-1})z\rho_1(z) &= \frac{1}{w - q^{-k_1-1}z} : e^{r_1^+(z)+a_2(w)} : - \frac{1}{w - q^{k_1+1}z} : e^{r_1^-(z)+a_2(w)} :. \end{aligned}$$

Upon integration and shifting contours, we obtain

$$\begin{aligned} (q - q^{-1})[\rho_1, \sigma_2]_+ &= \int \frac{dz}{z} \left( : e^{r_1^+(z)+a_2(q^{-k_1-1}z)} : - : e^{r_1^-(z)+a_2(q^{k_1+1}z)} : \right) \\ &= \int \frac{dz}{z} \left( : e^{r_1^+(q^{k_1+2}z)+a_2(qz)} : - : e^{r_1^-(q^{-k_1-2}z)+a_2(q^{-1}z)} : \right). \end{aligned}$$

Now it suffices to apply (3.8) which implies  $r_1^+(q^{k_1+2}z) + a_2(qz) = r_1^-(q^{-k_1-2}z) + a_2(q^{-1}z)$ .

The other cases can be verified similarly. □

In the following sections we shall explain the algebraic background of these formulas. We shall show that the deformed screening currents are closely connected with generators of quantum toroidal algebras, see Proposition 5.1 and discussions in Section 6.

4. QUANTUM TOROIDAL  $\mathfrak{gl}_2$ 

In this section we introduce actions of the quantum toroidal  $\mathfrak{gl}_2$  algebra  $\mathcal{E}_2(q_1, q_2, q_3)$  on the total Fock space  $\mathcal{H}$ . For that purpose, we start from the Wakimoto representation of the quantum affine algebra  $U_q \widehat{\mathfrak{sl}}_2 \subset \mathcal{E}_2(q_1, q_2, q_3)$ .

The Wakimoto representation is defined on the Fock space of three free fields [Sh]. Re-expressing them in terms of  $a_i(z)$ ,  $i = 1, 2, 3$ , we are led to a realization of the Wakimoto representation on our Fock space  $\mathcal{H}$ . Leaving the technical details to Appendix B, we present the result below.

Let us introduce fields  $u_1^\pm(z)$ ,  $v_1^\pm(z)$ ,

$$(4.1) \quad u_1^\pm(z) = Q_{u_1} + u_{1,0} \log z \mp a_{3,0} \log q - \sum_{n \neq 0} \frac{u_{1,n}^\pm}{n} z^{-n},$$

$$(4.2) \quad v_1^\pm(z) = -(Q_{u_1} + u_{1,0} \log z) \pm a_{2,0} \log q - \sum_{n \neq 0} \frac{v_{1,n}^\pm}{n} z^{-n}.$$

The oscillator part is given by

$$(4.3) \quad u_{1,n}^+ = \begin{cases} \frac{[n]}{[(k_3+2)n]} (q^{k_1 n} a_{1,n} + q^{-(k_2+2)n} a_{2,n}) & (n < 0), \\ -\frac{[n]}{[(k_2+2)n]} (a_{1,n} + q^{-(k_2+2)n} a_{3,n}) & (n > 0), \end{cases}$$

$$(4.4) \quad u_{1,n}^- = u_{1,n}^+ - (q^n - q^{-n}) a_{3,n},$$

$$(4.5) \quad v_{1,n}^- = \begin{cases} -\frac{[n]}{[(k_3+2)n]} (a_{1,n} + q^{-(k_3+2)n} a_{2,n}) & (n < 0), \\ \frac{[n]}{[(k_2+2)n]} (q^{k_1 n} a_{1,n} + q^{-(k_3+2)n} a_{3,n}) & (n > 0), \end{cases}$$

$$(4.6) \quad v_{1,n}^+ = v_{1,n}^- - (q^n - q^{-n}) a_{2,n}.$$

The zero mode part is

$$(4.7) \quad Q_{u_1} = Q_{u_1}^0 + s \bar{Q}_{u_1}, \quad u_{1,0} = u_{1,0}^0 + \frac{s}{1 + k_1 s} \bar{u}_{1,0},$$

with

$$(4.8) \quad Q_{u_1}^0 = \frac{1}{k_3 + 2} (Q_{a_1} + Q_{a_2}), \quad u_{1,0}^0 = -\frac{1}{k_2 + 2} (a_{1,0} + a_{3,0}),$$

$$(4.9) \quad \bar{Q}_{u_1} = \frac{1}{k_3 + 2} (k_1 Q_{a_1} - (k_2 + 2) Q_{a_2} - (k_3 + 2) Q_{a_3}),$$

$$(4.10) \quad \bar{u}_{1,0} = \frac{1}{k_2 + 2} (k_1 a_{1,0} - (k_2 + 2) a_{2,0} - (k_3 + 2) a_{3,0}).$$

Here  $s$  is an arbitrary parameter. We will make a specific choice later on, see (5.3).

Set further

$$(4.11) \quad h_{1,0} = -\frac{1}{1 + k_1 s} \bar{u}_{1,0},$$

$$(4.12) \quad h_{1,n} = \frac{1}{n} \left( [k_1 n] a_{1,n} - [(k_2 + 2)n] a_{2,n} - [(k_3 + 2)n] a_{3,n} \right) \times \begin{cases} -\frac{[n]}{[(k_2+2)n]} & (n > 0), \\ \frac{[n]}{[(k_3+2)n]} & (n < 0). \end{cases}$$

Let  $E_1(z), F_1(z), K_1^{\pm 1}, H_{1,n}$  ( $n \neq 0$ ) be the standard generators of  $U_q \widehat{\mathfrak{sl}}_2$  (for our convention see Appendix A).

**Proposition 4.1.** *The following assignment gives the Wakimoto representation of  $U_q \widehat{\mathfrak{sl}}_2$  of level  $q^{k_1}$ :*

$$(4.13) \quad -(q - q^{-1})E_1(z) \mapsto: e^{u_1^+(z)} : - : e^{u_1^-(z)} : , \quad (q - q^{-1})F_1(z) \mapsto: e^{v_1^+(z)} : - : e^{v_1^-(z)} : ,$$

$$(4.14) \quad K_1 \mapsto q^{h_{1,0}} , \quad H_{1,n} \mapsto h_{1,n} .$$

On the Fock space  $\mathcal{H}$  we have an extra boson consisting of  $\{a_{0,n}\}$ . The combination

$$Z_n = (q^n + q^{-n})a_{0,n} + \sum_{i=1}^3 (q^{(k_i+1)n} + q^{-(k_i+1)n})a_{i,n}$$

is a Heisenberg algebra commuting with  $\{a_{i,n}\}$ ,  $i = 1, 2, 3$ , and hence with the action of  $U_q \widehat{\mathfrak{sl}}_2$ . Altogether we have a representation of  $U_q \widehat{\mathfrak{gl}}_2$  on  $\mathcal{H}$ . This representation can be further lifted to one of  $\mathcal{E}_2(q_1, q_2, q_3)$  via the following evaluation morphism:

**Proposition 4.2.** [Mi] *Let  $\bar{\mathcal{E}}_2(q_1, q_2, q_3)$  denote the quotient of  $\mathcal{E}_2(q_1, q_2, q_3)$  by the relation  $C = q_3$ , where  $C$  denotes the central element. Then there exists a homomorphism of algebras  $\bar{\mathcal{E}}_2(q_1, q_2, q_3) \rightarrow U_q \widehat{\mathfrak{gl}}_2$ .*

We use the formulas in [FJM1] for the evaluation homomorphism to arrive at the following conclusion.

**Theorem 4.3.** *Let  $\mathcal{E}_{2,1}$  denote the quantum toroidal algebra  $\mathcal{E}_2(q_1, q_2, q_3)$  with parameters*

$$q_1 = q^{-k_1-2}, \quad q_2 = q^2, \quad q_3 = q^{k_1} .$$

*Then the following assignment along with (4.13), (4.14) gives a representation  $\pi_1 : \mathcal{E}_{2,1} \rightarrow \text{End } \mathcal{H}$ .*

$$(4.15) \quad -(q - q^{-1})E_0(z) \mapsto: e^{\hat{u}_1^+(z)} : - : e^{\hat{u}_1^-(z)} : , \quad (q - q^{-1})F_0(z) \mapsto: e^{\hat{v}_1^+(z)} : - : e^{\hat{v}_1^-(z)} : ,$$

$$(4.16) \quad K_0 \mapsto q^{-h_{1,0}}, \quad H_{0,n} \mapsto \hat{h}_{1,n} ,$$

where

$$(4.17) \quad \hat{u}_1^\pm(z) = -(Q_{u_1} + u_{1,0} \log z) \mp a_{2,0} \log q - \sum_{n \neq 0} \frac{\hat{u}_{1,n}^\pm}{n} z^{-n} ,$$

$$(4.18) \quad \hat{v}_1^\pm(z) = Q_{u_1} + u_{1,0} \log z \pm a_{3,0} \log q - \sum_{n \neq 0} \frac{\hat{v}_{1,n}^\pm}{n} z^{-n} ,$$

$$(4.19) \quad \hat{h}_{1,n} = \frac{1}{n} \left( [k_1 n] a_{0,n} - [(k_3 + 2)n] a_{2,n} - [(k_2 + 2)n] a_{3,n} \right) \times \begin{cases} -\frac{[n]}{[(k_2 + 2)n]} & (n > 0), \\ \frac{[n]}{[(k_3 + 2)n]} & (n < 0), \end{cases}$$

and

$$(4.20) \quad \hat{u}_{1,n}^+ = \begin{cases} \frac{[n]}{[(k_3 + 2)n]} \left( q^{k_1 n} a_{0,n} + q^{-(k_2 + 2)n} a_{3,n} \right) & (n < 0) \\ -\frac{[n]}{[(k_2 + 2)n]} \left( a_{0,n} + q^{-(k_2 + 2)n} a_{2,n} \right) & (n > 0) \end{cases}$$

$$(4.21) \quad \hat{u}_{1,n}^- = \hat{u}_{1,n}^+ - (q^n - q^{-n}) a_{2,n} ,$$

$$(4.22) \quad \hat{v}_{1,n}^- = \begin{cases} -\frac{[n]}{[(k_3 + 2)n]} \left( a_{0,n} + q^{-(k_3 + 2)n} a_{3,n} \right) & (n < 0) \\ \frac{[n]}{[(k_2 + 2)n]} \left( q^{k_1 n} a_{0,n} + q^{-(k_3 + 2)n} a_{2,n} \right) & (n > 0) \end{cases}$$

$$(4.23) \quad \hat{v}_{1,n}^+ = \hat{v}_{1,n}^- - (q^n - q^{-n}) a_{3,n} .$$

**Lemma 4.4.** *The bosonic screening  $\rho_1$  and the fermionic screenings  $\sigma_2, \sigma_3$  (anti-)commute with the action of  $\mathcal{E}_{2,1}$ .*



*Proof.* This can be verified in the same way as in Proposition 3.1. We use Table 3 and the identities

$$\begin{aligned} u_1^+(z) - u_1^-(z) &= -a_3(qz) + a_3(q^{-1}z), \\ v_1^+(z) - v_1^-(z) &= -a_2(q^{-1}z) + a_2(qz), \\ \hat{u}_1^+(z) - \hat{u}_1^-(z) &= -a_2(qz) + a_2(q^{-1}z), \\ \hat{v}_1^+(z) - \hat{v}_1^-(z) &= -a_3(q^{-1}z) + a_3(qz). \end{aligned}$$

□

Cyclically permuting the indices  $(1, 2, 3)$ , we obtain three different actions of quantum toroidal  $\mathfrak{gl}_2$  algebras on  $\mathcal{H}$ .

**Corollary 4.5.** *Let  $\mathcal{E}_{2,i} = \mathcal{E}_2(q^{-k_i-2}, q^2, q^{k_i})$ ,  $i = 1, 2, 3$ . Then there exist representations  $\pi_i : \mathcal{E}_{2,i} \rightarrow \text{End } \mathcal{H}$ .*

We are not aware of any reasonable relations between algebras  $\mathcal{E}_{2,i}$  with different  $i$ . These actions do not mutually commute, in fact even the quantum affine subalgebras do not. The situation is different from the quantum toroidal  $(\mathfrak{gl}_m, \mathfrak{gl}_n)$  duality [FJM2] where the quantum toroidal algebras do not commute but the quantum affine subalgebras mutually commute.

## 5. INTEGRALS OF MOTION

In [FJM] it was shown that the standard transfer matrix construction provides a commutative subalgebra inside a completion of  $\mathcal{E}_2$ . By “transfer matrix” we mean a weighted trace of the universal R matrix of  $\mathcal{E}_2$ . In the course of taking trace, one is naturally led to consider dressed currents

$$\begin{aligned} F_i^{dress}(z, p) &= F_i(z) \prod_{\ell \geq 0} \bar{K}_i^+(p^\ell z)^{-1} \\ &= F_i(z) \exp\left(-(q - q^{-1}) \sum_{n > 0} \frac{1}{1 - p^n} H_{i,n} z^{-n}\right). \end{aligned}$$

The Taylor coefficients of the transfer matrix are then multiple integrals involving products of the dressed currents, giving the deformed analog of the “non-local integrals of motion”. The aim of this section is to discuss integrals of motion

We begin by observing that the bosonic screening currents  $\rho_i(z), \tau_i(z)$  are the images of these dressed currents on the Fock space  $\mathcal{H}$ , modulo a simple modification in the zero mode. In the following we set

$$p_i = q^{2(k_i+2)}, \quad i = 1, 2, 3,$$

and assume that  $|p_i| < 1$  for  $i = 1, 2, 3$ . This implies in particular that  $|q^4| = |p_1 p_2 p_3| < 1$ .

**Proposition 5.1.** *We have the relations*

$$\begin{aligned} z\rho_3(z) &= e^{s\bar{Q}_{u_1}} \pi_1\left(F_1^{dress}(z, p_3)\right) z^{-\lambda h_{1,0}}, \\ z\tau_3(z) &= e^{-s\bar{Q}_{u_1}} \pi_1\left(F_0^{dress}(z, p_3)\right) z^{\lambda h_{1,0}}, \end{aligned}$$

where  $\bar{Q}_{u_1}$  is given in (4.9) and

$$\lambda = \frac{1 - (k_2 + 2)s}{k_3 + 2}.$$

*Proof.* This follows from the identities

$$\begin{aligned} r_3^\pm(z) - v_1^\pm(z) &= s\bar{Q}_{u_1} - \lambda h_{1,0} \log z - (q - q^{-1}) \sum_{n>0} \frac{1}{1 - p_3^n} h_{1,n} z^{-n}, \\ t_3^\pm(z) - \hat{v}_1^\pm(z) &= -s\bar{Q}_{u_1} + \lambda h_{1,0} \log z - (q - q^{-1}) \sum_{n>0} \frac{1}{1 - p_3^n} \hat{h}_{1,n} z^{-n}, \end{aligned}$$

which can be checked directly using (4.1)–(4.12), (4.17)–(4.23) and (4.7).  $\square$

Proposition 5.1 allows us to rewrite

$$\begin{aligned} & \prod_{1 \leq j \leq N}^{\curvearrowright} \pi_1(F_0^{\text{dress}}(x_{0,j}, p_3)) \prod_{1 \leq j \leq N}^{\curvearrowright} \pi_1(F_1^{\text{dress}}(x_{1,j}, p_3)) \\ &= \prod_{1 \leq j \leq N}^{\curvearrowright} x_{0,j} \tau_3(x_{0,j}) \prod_{1 \leq j \leq N}^{\curvearrowright} x_{1,j} \rho_3(x_{1,j}) \times \prod_{j=1}^N x_{0,j}^{\frac{2j}{k_3+2}} x_{1,j}^{-\frac{2(N-j)}{k_3+2}} \prod_{j=1}^N \left( \frac{x_{1,j}}{x_{0,j}} \right)^{\lambda h_{1,0}}, \end{aligned}$$

where  $\prod_{1 \leq j \leq N}^{\curvearrowright} A_j = A_1 A_2 \cdots A_N$ . Inserting this into the known expressions for the integrals of motion associated with the quantum toroidal  $\mathfrak{gl}_2$  algebra [FJM] we obtain the following formula<sup>1</sup>

$$(5.1) \quad \mathbb{G}_{1,N}^{(\nu)} = \int \cdots \int \prod_{\substack{1 \leq j \leq N \\ t=0,1}} \frac{dx_{t,j}}{x_{t,j}} \prod_{1 \leq j \leq N}^{\curvearrowright} x_{0,j} \tau_3(x_{0,j}) \prod_{1 \leq j \leq N}^{\curvearrowright} x_{1,j} \rho_3(x_{1,j}) \cdot K_{1,N}^{(\nu)}(\mathbf{x}; p_3),$$

where  $\mathbf{x} = (x_{0,1}, \dots, x_{0,N}, x_{1,1}, \dots, x_{1,N})$ ,  $\nu = 0, 1$ ,

$$\begin{aligned} (5.2) \quad K_{1,N}^{(\nu)}(\mathbf{x}; p) &= \frac{\prod_{t=0,1} \prod_{j < l} \Theta_p(x_{t,l}/x_{t,j}, q^2 x_{t,l}/x_{t,j})}{\prod_{j,l} \Theta_p(q^{k_1+2} x_{1,l}/x_{0,j}, q^{-k_1} x_{1,l}/x_{0,j})} \vartheta_\nu \left( q^{h_{1,0}/2} \prod_{j=1}^N \frac{x_{1,j}}{x_{0,j}}, p \right) \\ &\times \prod_{j=1}^N x_{0,j}^{1-2j+\frac{2j}{k_3+2}} x_{1,j}^{1-2(N-j)-\frac{2(N-j)}{k_3+2}} \left( \prod_{j=1}^N \frac{x_{1,j}}{x_{0,j}} \right)^{\lambda h_{1,0}}, \end{aligned}$$

and  $\Theta_p(z) = \prod_{n=1}^\infty (1 - p^{n-1}z)(1 - p^n z^{-1})(1 - p^n)$ ,  $\vartheta_\nu(z, p) = \sum_{n \in \mathbb{Z} + \nu/2} p^{n^2} z^{2n}$ . The contours of integration are deformations of the unit circle  $|x_{0,j}| = |x_{1,j}| = 1$  from the domain where the parameters satisfy  $|q^{-k_1-2}| < 1, |q^{k_1}| < 1$ . More specifically, for each  $i = 0, 1$  and  $j = 1, \dots, N$ , the  $x_{i,j}$  integration contour is such that the series of poles

$$p_3^m q^{-k_1-2} x_{1-i,l}, \quad p_3^m q^{k_1} x_{1-i,l} \quad (1 \leq l \leq N, m \geq 0)$$

are inside and

$$p_3^{-m} q^{k_1+2} x_{1-i,l}, \quad p_3^{-m} q^{-k_1} x_{1-i,l} \quad (1 \leq l \leq N, m \geq 0)$$

are outside.

From now on, we choose

$$(5.3) \quad s = \frac{1}{2(k_2 + 2)}$$

so that  $\lambda = \frac{1}{2(k_3+2)}$ . The point of the choice (5.3) is to make the kernel function (5.2) quasi-periodic, having a factor of quasi-periodicity independent of  $h_{1,0}$ . That is,

<sup>1</sup>We use [FJM], eq.(3.11), choosing  $\bar{p}_1 = 1$  in the argument of  $\vartheta_\nu$ . Also the factor  $q^{-H_{1,0}/2}$  there should be corrected to  $q^{H_{1,0}/2}$ .

**Lemma 5.2.** *We have*

$$K_{1,N}^{(\nu)}(\mathbf{x}; p) \big|_{x_{0,j} \rightarrow px_{0,j}} = q^2 K_{1,N}^{(\nu)}(\mathbf{x}; p), \quad K_{1,N}^{(\nu)}(\mathbf{x}; p) \big|_{x_{1,j} \rightarrow px_{1,j}} = q^2 K_{1,N}^{(\nu)}(\mathbf{x}; p).$$

*Proof.* This can be checked directly using  $\Theta_p(pz) = -z^{-1}\Theta_p(z)$ ,  $\vartheta_\nu(pz; p) = p^{-1}z^{-2}\vartheta_\nu(z; p)$ .  $\square$

With the choice (5.3), and after normal ordering, the integrand of (5.1) comprises only integer powers except for the factor  $(\prod_{j=1}^N x_{1,j}/x_{0,1})^{u_{1,0}}$ , where  $u_{1,0}$  given in (4.7) simplifies to

$$u_{1,0} = \frac{1}{k_3 - k_2} (2a_{1,0} + a_{2,0} + a_{3,0}).$$

Hence the action of  $U_q \widehat{\mathfrak{sl}}_2 \subset \mathcal{E}_{2,1}$  and  $\mathbb{G}_{1,N}^{(\nu)}$  are defined on sectors on which  $2a_{1,0} + a_{2,0} + a_{3,0}$  has eigenvalues in  $(k_3 - k_2)\mathbb{Z}$ .

Cyclically permuting the indices  $(1, 2, 3)$  we obtain three kinds of integrals of motion  $\mathbb{G}_{i,N}^{(\nu)}$  associated with  $\mathcal{E}_{2,i}$ ,  $i = 1, 2, 3$ . These operators are well-defined on sectors  $\mathcal{H}_{N_1, N_2, N_3}$  ( $N_i \in \mathbb{Z}$ ) on which  $a_{i,0}$  have eigenvalues

$$a_{1,0} = \frac{3}{4}(k_2 - k_3)N_1 - \frac{1}{4}(k_3 - k_1)N_2 - \frac{1}{4}(k_1 - k_2)N_3 \quad \& \text{ cycl..}$$

Even though the algebras  $\mathcal{E}_{2,i}$  do not commute with each other, the corresponding integrals of motion do. The following “trality” is the main result of this paper.

**Theorem 5.3.** *On sectors  $\mathcal{H}_{N_1, N_2, N_3}$ , the three kinds of integrals of motion are mutually commutative. Namely we have*

$$[\mathbb{G}_{i,M}^{(\mu)}, \mathbb{G}_{j,N}^{(\nu)}] = 0 \quad (i, j = 1, 2, 3, M, N \geq 1, \mu, \nu = 0, 1).$$

*Proof.* When  $i = j$  the commutativity is known (and is evident by construction). So it suffices to consider  $i = 1, j = 2$ . The product  $\mathbb{G}_{1,M}^{(\mu)} \mathbb{G}_{2,N}^{(\nu)}$  is an integral

$$\begin{aligned} & \int \cdots \int \prod_{\substack{1 \leq j \leq M \\ t=0,1}} \frac{dx_{t,j}}{x_{t,j}} \prod_{\substack{1 \leq l \leq N \\ t=0,1}} \frac{dy_{t,l}}{y_{t,l}} K_{1,M}^{(\mu)}(\mathbf{x}; p_3) K_{2,N}^{(\nu)}(\mathbf{y}; p_1) \\ & \times \prod_{1 \leq j \leq M}^{\curvearrowright} x_{0,j} \tau_3(x_{0,j}) \prod_{1 \leq j \leq M}^{\curvearrowright} x_{1,j} \rho_3(x_{1,j}) \prod_{1 \leq l \leq N}^{\curvearrowright} y_{0,l} \tau_1(y_{0,l}) \prod_{1 \leq l \leq N}^{\curvearrowright} y_{1,l} \rho_1(y_{1,l}). \end{aligned}$$

From Table 2, contractions of  $\tau_3(x_{0,j})$  or  $\rho_3(x_{1,j})$  with  $\tau_1(y_{0,l})$  or  $\rho_1(y_{1,l})$  are rational functions with simple poles. The commutator  $[\mathbb{G}_{1,M}^{(\mu)}, \mathbb{G}_{2,N}^{(\nu)}]$  is a sum of terms obtained by taking residues from each such contraction.

Let us use  $z, w$  to represent variables that enter the relevant contraction. There are four cases to consider,

$$\begin{aligned} \rho_3^\pm(z) \rho_1^\pm(w) &= q^{\mp 1} \frac{1 - q^{\mp k_1} w/z}{1 - q^{\mp(k_1+2)} w/z} : e^{r_3^\pm(z) + r_1^\pm(w)} : + \cdots, \\ \rho_3^\pm(z) \tau_1^\pm(w) &= q^{\pm 1} \frac{1 - q^{\pm k_3} w/z}{1 - q^{\pm(k_3+2)} w/z} : e^{r_3^\pm(z) + t_1^\pm(w)} : + \cdots, \\ \tau_3^\mp(z) \rho_1^\pm(w) &= q^{\pm 1} \frac{1 - q^{\mp 2} w/z}{1 - w/z} : e^{t_3^\mp(z) + r_1^\pm(w)} : + \cdots, \\ \tau_3^\mp(z) \tau_1^\pm(w) &= q^{\mp 1} \frac{1 - q^{\pm(k_2+2)} w/z}{1 - q^{\pm k_2} w/z} : e^{t_3^\mp(z) + t_1^\pm(w)} : + \cdots. \end{aligned}$$

In the first three cases, integration with respect to  $z$  yields the same term with opposite signs, due to the identities which follow from (3.7)–(3.10):

$$\begin{aligned} r_3^+(w) - r_3^-(w) &= -r_1^+(q^{k_1+2}w) + r_1^-(q^{-k_1-2}w), \\ r_3^+(q^{k_3+2}w) - r_3^-(q^{-k_3-2}w) &= -t_1^+(w) + t_1^-(w), \\ t_3^+(w) - t_3^-(w) &= r_1^+(w) - r_1^-(w). \end{aligned}$$

Therefore the residues sum up to zero. In the last case we need a shift of arguments. Noting that  $p_1 q^{k_2} = p_3^{-1} q^{-k_2}$ , we have

$$\begin{aligned} & (t_3^-(q^{k_2}w) + t_1^+(w))|_{w \rightarrow p_1 w} - (t_3^+(q^{-k_2}w) + t_1^-(w)) \\ &= -(t_3^+(q^{-k_2}w) - t_3^-(p_3^{-1} q^{-k_2}w)) + (t_1^+(p_1 w) - t_1^-(w)) = 0. \end{aligned}$$

Thanks to Lemma 5.2, the product of the kernel functions remains unchanged under this shift,

$$K_{3,M}^{(\mu)}(\mathbf{x}; p_3)|_{x_{j,0} \rightarrow p_3^{-1} x_{j,0}} K_{1,N}^{(\nu)}(\mathbf{y}; p_1)|_{y_{l,0} \rightarrow p_1 y_{l,0}} = K_{3,M}^{(\mu)}(\mathbf{x}; p_3) K_{1,N}^{(\nu)}(\mathbf{y}; p_1).$$

Case by case analysis shows that the contours also match after shift.  $\square$

## 6. $D(2, 1; \alpha)$ STRUCTURE

Finally we make some comments regarding the deformed fermionic screening currents and the possible connection with the “quantum toroidal  $D(2, 1; \alpha)$ ” algebra.

Let us begin with the affine superalgebra  $\hat{D}(2, 1; \alpha)$  with  $\alpha = k_1/(k_3 + 2)$ . We choose all simple roots  $\alpha_i$  to be fermionic, that is  $(\alpha_i, \alpha_i) = 0$ , so that the Dynkin diagram is as in figure 1 below. The quantum affine superalgebra  $U_{\mathbf{q}} \hat{D}(2, 1; \alpha)$  has been studied in [HSTY], where the Drinfeld realization is obtained. Here we take the deformation parameter as  $\mathbf{q} = q^{-k_3-2}$ .

We are interested only in the Borel subalgebra of  $U_{\mathbf{q}} \hat{D}(2, 1; \alpha)$ . The latter is generated by commutative elements  $\mathbf{h}_{i,n}$  and fermionic currents  $\xi_i(z)$  ( $i = 1, 2, 3$ ,  $n > 0$ ), satisfying

$$(6.1) \quad [\mathbf{h}_{i,n}, \xi_j(z)] = \frac{[(\alpha_i, \alpha_j)n]}{n} z^n \xi_j(z),$$

$$(6.2) \quad (z - q^{(\alpha_i, \alpha_j)} w) \xi_i(z) \xi_j(w) = (w - q^{(\alpha_i, \alpha_j)} z) \xi_j(w) \xi_i(z),$$

$$(6.3) \quad [(\alpha_1, \alpha_3)] [[\xi_1(z_1), \xi_2(z_2)], \xi_3(z_3)] = [(\alpha_1, \alpha_2)] [[\xi_1(z_1), \xi_3(z_3)], \xi_2(z_2)] \quad \& \text{ cycl.}$$

Here, for elements  $X, Y$  of parity  $\bar{X}, \bar{Y} \in \{0, 1\}$  and weight  $\text{wt } X, \text{wt } Y$  we set

$$[[X, Y]] = XY - (-)^{\bar{X}\bar{Y}} q^{-(\text{wt } X, \text{wt } Y)} YX.$$

To our knowledge, quantum toroidal version of the  $D(2, 1; \alpha)$  algebra has not been defined in the literature. Guided by the usual construction of quantum toroidal algebras, we consider an algebra  $\mathbf{B}$  generated by symbols  $\mathbf{h}_{i,n}$ ,  $\xi_i(z)$  for  $i = 0, 1, 2, 3$  satisfying (6.1)–(6.3) for all  $i, j, k \in \{0, 1, 2, 3\}$ . We take  $\mathbf{B}$  as a working definition of the Borel part of the quantum toroidal  $D(2, 1; \alpha)$  algebra.

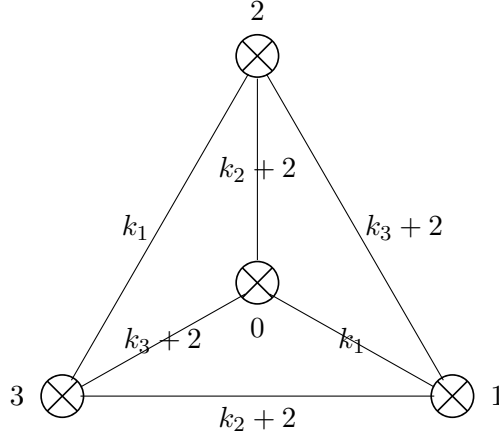


FIGURE 1. Dynkin diagram of  $\hat{D}(2,1;\alpha)$ ,  $\alpha = \frac{k_1}{k_3+2}$ . The numbers inscribed on the edges indicate the scalar product of roots, e.g.,  $(\alpha_0, \alpha_1) = k_1$ .

It is not hard to find a free field realization of  $\mathbf{B}$ . Let

$$\varepsilon_i = \begin{cases} 1 & i = 0, 1, \\ -1 & i = 2, 3, \end{cases}$$

and define matrices  $A, B$  by

$$A_{i,j} = \varepsilon_i \varepsilon_j q^{-(\alpha_i, \alpha_j) \varepsilon_i \varepsilon_j n}, \quad B_{i,j} = q^{(\alpha_i, \alpha_j) n} - q^{-(\alpha_i, \alpha_j) n}, \quad i, j \in \{0, 1, 2, 3\}.$$

Introducing one pair of zero mode  $Q, a$  with  $[a, Q] = 1$ , we consider fields

$$\mathbf{x}_i(z) = \varepsilon_i (Q + a \log z) - \sum_{n \neq 0} \frac{\mathbf{x}_{i,n}}{n} z^{-n}, \quad i = 0, 1, 2, 3,$$

where the oscillators  $\mathbf{x}_{i,n}$  satisfy

$$[\mathbf{x}_{i,n}, \mathbf{x}_{j,-n}] = n A_{i,j}.$$

Then the following assignment gives a realization of (6.1)–(6.3) with  $i = 0, 1, 2, 3$ .

$$\xi_i(z) \mapsto e^{\mathbf{x}_i(z)},$$

$$(q - q^{-1}) n \mathbf{h}_{i,n} \mapsto \sum_{j=0}^3 (B A^{-1})_{i,j} \mathbf{x}_{j,n}.$$

The fermionic fields  $a_i(z)$  considered in the previous sections can be viewed as currents dressed from  $\mathbf{x}_i(z)$  by  $\mathbf{h}_{i,n}$ . Indeed, setting

$$\mathbf{a}_{i,n} = \mathbf{x}_{i,n} - (q - q^{-1}) \frac{q^{2n}}{1 - q^{2n}} n \mathbf{h}_{i,n},$$

$$\mathbf{a}_{i,-n} = \mathbf{x}_{i,-n},$$

for  $n > 0$ , we obtain

$$[\mathbf{a}_{i,n}, \mathbf{a}_{j,-n}] = n \frac{[(\alpha_i, \alpha_j) + \varepsilon_i \varepsilon_j] n}{[n]}.$$

These are exactly the relations (3.1), (3.3), (3.2) for the oscillator part of  $a_i(z)$ .

The quantum toroidal  $D(2, 1; \alpha)$  algebra and its commutative subalgebra warrant further investigation. For that purpose the shuffle algebra approach might be useful. We hope to return to this subject in the future.

#### APPENDIX A. QUANTUM TOROIDAL $\mathfrak{gl}_2$ ALGEBRA

We give here the defining relations of quantum toroidal  $\mathfrak{gl}_2$  algebra  $\mathcal{E}_2 = \mathcal{E}_2(q_1, q_2, q_3)$ .

Let  $(a_{i,j}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$  be the Cartan matrix of type  $A_1^{(1)}$ . Define  $q, d$  by  $q_2 = q^2$ ,  $q_1 = d/q$  and  $q_3 = 1/(dq)$ . For  $r \neq 0$  we set

$$a_{i,j}(r) = \frac{[r]}{r} \times \begin{cases} q^r + q^{-r} & (i \equiv j \pmod{2}), \\ -d^r - d^{-r} & (i \not\equiv j \pmod{2}). \end{cases}$$

We set further

$$g_{i,j}(z, w) = \begin{cases} z - q_2 w & (i \equiv j \pmod{2}), \\ (z - q_1 w)(z - q_3 w) & (i \not\equiv j \pmod{2}). \end{cases}$$

Algebra  $\mathcal{E}_2(q_1, q_2, q_3)$  is generated by elements  $E_{i,k}, F_{i,k}, H_{i,r}$  and invertible elements  $K_i, C$ , where  $i = 0, 1$ ,  $k \in \mathbb{Z}$ ,  $r \in \mathbb{Z} \setminus \{0\}$ . We present below the defining relations in terms of generating series

$$\begin{aligned} E_i(z) &= \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k}, \\ K_i^\pm(z) &= K_i^{\pm 1} \bar{K}_i^\pm(z), \quad \bar{K}_i^\pm(z) = \exp\left(\pm(q - q^{-1}) \sum_{r>0} H_{i,\pm r} z^{\mp r}\right). \end{aligned}$$

The relations are as follows.

#### $C, K$ relations

$$\begin{aligned} C &\text{ is central, } K_0 K_1 = 1. \\ K_i E_j(z) K_i^{-1} &= q^{a_{i,j}} E_j(z), \quad K_i F_j(z) K_i^{-1} = q^{-a_{i,j}} F_j(z). \end{aligned}$$

#### $H$ - $H$ , $H$ - $E$ and $H$ - $F$ relations

$$\begin{aligned} [H_{i,r}, E_j(z)] &= a_{i,j}(r) C^{-(r+|r|)/2} z^r E_j(z), \\ [H_{i,r}, F_j(z)] &= -a_{i,j}(r) C^{-(r-|r|)/2} z^r F_j(z), \\ [H_{i,r}, H_{j,s}] &= \delta_{r+s,0} a_{i,j}(r) \frac{C^r - C^{-r}}{q - q^{-1}}. \end{aligned}$$

#### $E$ - $F$ relations

$$[E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} (\delta(C \frac{w}{z}) K_i^+(w) - \delta(C \frac{z}{w}) K_i^-(z)).$$

#### $E$ - $E$ and $F$ - $F$ relations

$$\begin{aligned} (-1)^{i+j} g_{i,j}(z, w) E_i(z) E_j(w) + g_{j,i}(w, z) E_j(w) E_i(z) &= 0, \\ (-1)^{i+j} g_{j,i}(w, z) F_i(z) F_j(w) + g_{i,j}(z, w) F_j(w) F_i(z) &= 0. \end{aligned}$$

**Serre relations** For  $i \neq j$ ,

$$\begin{aligned} \text{Sym}_{z_1, z_2, z_3} [E_i(z_1), [E_i(z_2), [E_i(z_3), E_j(w)]_{q^2}]]_{q^{-2}} &= 0, \\ \text{Sym}_{z_1, z_2, z_3} [F_i(z_1), [F_i(z_2), [F_i(z_3), F_j(w)]_{q^2}]]_{q^{-2}} &= 0. \end{aligned}$$

In the above we set  $[A, B]_p = AB - pBA$  and

$$\text{Sym } f(x_1, \dots, x_N) = \frac{1}{N!} \sum_{\pi \in \mathfrak{S}_N} f(x_{\pi(1)}, \dots, x_{\pi(N)}).$$

We identify the quantum affine algebra  $U_q \widehat{\mathfrak{sl}}_2$  with the subalgebra of  $\mathcal{E}_2(q_1, q_2, q_3)$  generated by  $E_1(z), F_1(z), H_{1,r}$  and  $K_1^{\pm 1}, C^{\pm 1}$ .

## APPENDIX B. COMPARISON WITH WAKIMOTO REPRESENTATION

In this section we explain the connection between the representation of  $\mathcal{E}_2$  in section 4 and the Wakimoto representation of  $U_q \widehat{\mathfrak{sl}}_2$ .

In [Sh], the Wakimoto representation of level  $q^k$  is written in terms of the generators

$$\begin{aligned} zJ^+(z) &= E_1(q^{-k-2}z), \quad zJ^-(z) = F_1(q^{-k-2}z), \\ J_n^3 &= q^{kn/2} H_{1,n}, \quad \psi(z) = K_1^+(q^{-k/2}z), \quad \varphi(z) = K_1^-(q^{-k/2}z), \end{aligned}$$

using three free fields  $a(z), b(z), c(z)$ . Non-trivial commutation relations among the Fourier modes of the latter are

$$(B.1) \quad [a_0, Q_a] = 2(k+2), \quad -[b_0, Q_b] = [c_0, Q_c] = 4,$$

$$(B.2) \quad [a_n, a_{-n}] = \frac{[2n][(k+2)n]}{n}, \quad -[b_n, b_{-n}] = [c_n, c_{-n}] = \frac{[2n]^2}{n}.$$

For  $x = a, b, c$  we set

$$\begin{aligned} x(L; M, N|z; \alpha) &= \frac{L}{MN} Q_x + \frac{L}{MN} x_0 \log z - \sum_{n \neq 0} \frac{[Ln]}{[Mn][Nn]} x_n z^{-n} q^{\alpha|n|}, \\ x(N|z; \alpha) &= x(L; L, N|z; \alpha). \end{aligned}$$

**Theorem B.1.** [Sh] *The following formula gives the Wakimoto representation of  $U_q \widehat{\mathfrak{sl}}_2$  of level  $q^k$ .*

$$\begin{aligned} -(q - q^{-1})zJ^+(z) &=: e^{\tilde{u}^+(z)} : - : e^{\tilde{u}^-(z)} :, \\ (q - q^{-1})zJ^-(z) &=: e^{\tilde{v}^+(z)} : - : e^{\tilde{v}^-(z)} :, \\ \psi(z) &= q^{a_0+b_0} e^{(q-q^{-1})\sum_{n>0} J_n^3 z^{-n}}, \\ \varphi(z) &= q^{-a_0-b_0} e^{-(q-q^{-1})\sum_{n>0} J_{-n}^3 z^n}, \end{aligned}$$

where

$$\begin{aligned} J_n^3 &= q^{2n-|n|} a_n + q^{(k+2)n - \frac{k+2}{2}|n|} b_n, \\ \tilde{u}^\pm(q^{k+2}z) &= -b(2|z; 1) - c(2|q^{\pm 1}z; 0), \\ \tilde{v}^\pm(q^{k+2}z) &= a(k+2|q^{k+2\pm k}z; -\frac{k+2}{2}) - a(k+2|q^kz; \frac{k+2}{2}) + b(2|q^{\pm(k+2)}z; -1) + c(2|q^{\pm(k+1)}z; 0). \end{aligned}$$

In addition the Wakimoto screening current is given by

$$\begin{aligned} -(q - q^{-1})zJ^S(z) &=: e^{\tilde{r}^+(z)} : - : e^{\tilde{r}^-(z)} : , \\ \tilde{r}^\pm(q^{k+2}z) &= -a(k+2|q^kz; -\frac{k+2}{2}) - b(2|z; -1) - c(2|q^{\pm 1}z; 0) . \end{aligned}$$

□

It is also straightforward to find two fermionic screening currents for  $U_{q\widehat{\mathfrak{sl}}_2}$

$$\begin{aligned} \tilde{\sigma}_2(z) &=: e^{a(1|q^{-2}z; -\frac{k+2}{2}) + b(k+2; 1, 2|q^{-k-2}z; -1) + c(k+1; 1, 2|q^{-k-2}z; 0)} : , \\ \tilde{\sigma}_3(z) &=: e^{c(2|q^{-k-2}z; 0)} : . \end{aligned}$$

Now we take  $k = k_1$ . Our task is to relate the free fields  $a(z), b(z), c(z)$  with  $a_1(z), a_2(z), a_3(z)$ . To this end we identify  $\sigma_i(z)$  with  $\tilde{\sigma}_i(q^{k+2}z)$ ,  $i = 2, 3$ . Comparison with the CFT case suggests that they commute with the Heisenberg subalgebra spanned by  $H_{1,n}$ , i.e.,  $[H_{1,n}, a_{2,m}] = [H_{1,n}, a_{3,m}] = 0$ . This fixes  $H_{1,n}$  uniquely up to a scalar multiple. Taking the normalization into account we obtain  $H_{1,n} = h_{1,n}$  where

$$(B.3) \quad h_{1,n} = \frac{1}{n} \left( [k_1 n] a_{1,n} - [(k_2 + 2)n] a_{2,n} - [(k_3 + 2)n] a_{3,n} \right) \times \begin{cases} -\frac{[n]}{[(k_2 + 2)n]} & (n > 0), \\ \frac{[n]}{[(k_3 + 2)n]} & (n < 0). \end{cases}$$

The CFT case suggests also that we compare the bosonic screening current  $\rho_1(z)$  with the Wakimoto screening current  $-J^S(q^{k+2}z)$ . Equating the oscillator part and using the relation  $J_n^3 = q^{kn/2}h_{1,n}$ , we can solve  $a_n, b_n, c_n$  ( $n \neq 0$ ) for  $h_{1,n}, a_{2,n}, a_{3,n}$  as follows.

$$(B.4) \quad nq^{-kn + \frac{k+2}{2}|n|} a_n = \frac{[(k+2)n]}{[kn]} q^{\frac{k}{2}(|n|-n)-2n} n h_{1,n} - \frac{[n][2n]}{[kn]} a_{2,n} + \frac{[(k+1)n][2n]}{[kn]} a_{3,n} ,$$

$$(B.5) \quad nq^{|n|} b_n = -\frac{[2n]}{[kn]} q^{-\frac{k}{2}(n+|n|)-2n} n h_{1,n} + \frac{[n][2n]}{[kn]} a_{2,n} - \frac{[(k+1)n][2n]}{[kn]} a_{3,n} ,$$

$$(B.6) \quad n c_n = [2n] a_{3,n} .$$

From the zero mode of  $\sigma_i(z)$  we have

$$\begin{aligned} \frac{1}{2}Q_c &= Q_{a_3} , \quad Q_a + \frac{k+2}{2}Q_b + \frac{k+1}{2}Q_c = Q_{a_2} , \\ \frac{1}{2}c_0 &= a_{3,0} , \quad a_0 + \frac{k+2}{2}b_0 + \frac{k+1}{2}c_0 = a_{2,0} . \end{aligned}$$

Thus we get  $-(q - q^{-1})E_1(z) =: e^{u_1^+(z)} : - : e^{u_1^-(z)} :$ ,  $(q - q^{-1})F_1(z) =: e^{v_1^+(z)} : - : e^{v_1^-(z)} :$  with

$$\begin{aligned} u_1^\pm(z) &= \tilde{u}^\pm(q^{k+2}z) = Q_{u_1} + u_{1,0} \log z \mp a_{3,0} \log q - \sum_{n \neq 0} \frac{u_{1,n}^\pm}{n} z^{-n} , \\ v_1^\pm(z) &= \tilde{v}^\pm(q^{k+2}z) = -(Q_{u_1} + u_{1,0} \log z) \pm a_{2,0} \log q - \sum_{n \neq 0} \frac{v_{1,n}^\pm}{n} z^{-n} . \end{aligned}$$

Formulas for  $u_{1,n}^\pm, v_{1,n}^\pm$  are obtained by using (B.3) and (B.4)–(B.6). The result is given in (4.1) and (4.2).

It remains to express the zero mode

$$Q_{u_1} = -\frac{1}{2}(Q_b + Q_c), \quad u_{1,0} = -\frac{1}{2}(b_0 + c_0) ,$$



in terms of  $Q_{a_i}$  and  $a_{i,0}$ . We have

$$\begin{aligned} [u_{1,0}, Q_{u_1}] &= 0, & [u_{1,0}, Q_{a_2}] &= 1, & [u_{1,0}, Q_{a_3}] &= -1, \\ [a_{2,0}, Q_{u_1}] &= 1, & [a_{3,0}, Q_{u_1}] &= -1. \end{aligned}$$

These equations admit a one-parameter family of solutions (4.7), (4.8)–(4.10) given in the main text.

For any choice of  $s$ , we have an identification

$$\begin{aligned} Q_a &= (k_1 + 2)Q_{u_1} + Q_{a_2} + Q_{a_3}, & Q_b &= -2Q_{u_1} - 2Q_{a_3}, & Q_c &= 2Q_{a_3}, \\ a_0 &= (k_1 + 2)u_{1,0} + a_{2,0} + a_{3,0}, & b_0 &= -2u_{1,0} - 2a_{3,0}, & c_0 &= 2a_{3,0}, \\ h_{1,0} &= a_0 + b_0 = -\frac{1}{1 + k_1 s} \bar{u}_{1,0}. \end{aligned}$$

Finally the bosonic screening current is related to the Wakimoto screening current by

$$\rho_1(q^{-k-2}z) = -J^S(z) q^{\frac{k}{k+2}a_0}.$$

The factor  $a_0$  does not affect the screening property because it commutes with  $J^\pm(z), J^3(z)$ .

## APPENDIX C. CONTRACTIONS

In this section we summarize contractions among various currents used in the text. We write

$$\rho_j^\pm(z) =: e^{r_j^\pm(z)} :, \quad \tau_j^\pm(z) =: e^{t_j^\pm(z)} :.$$

In the following all relations hold true under simultaneous cyclic permutations of indices  $(1, 2, 3)$ , so we write only typical cases and omit writing “& *cycl.*”.

*Contractions for  $\sigma_i(z)$*

$$\begin{aligned} \sigma_i(z)\sigma_i(w) &= (z - w) : e^{a_i(z) + a_i(w)} :, \\ \sigma_0(z)\sigma_1(w) &= z^{k_1+1} \frac{(q^{-k_1}w/z; q^2)_\infty}{(q^{k_1+2}w/z; q^2)_\infty} : e^{a_0(z) + a_1(w)} :, \\ \sigma_1(w)\sigma_0(z) &= w^{k_1+1} \frac{(q^{-k_1}z/w; q^2)_\infty}{(q^{k_1+2}z/w; q^2)_\infty} : e^{a_0(z) + a_1(w)} :, \\ \sigma_2(z)\sigma_3(w) &= z^{k_1+1} \frac{(q^{-k_1}w/z; q^2)_\infty}{(q^{k_1+2}w/z; q^2)_\infty} : e^{a_2(z) + a_3(w)} :, \\ \sigma_3(w)\sigma_2(z) &= w^{k_1+1} \frac{(q^{-k_1}z/w; q^2)_\infty}{(q^{k_1+2}z/w; q^2)_\infty} : e^{a_2(z) + a_3(w)} :. \end{aligned}$$

*Contractions for  $\sigma_i(z)$  and  $\rho_j^\pm(z), \tau_j^\pm(z)$*  In all cases we have equalities (in the sense of matrix elements)

$$\rho_j^\pm(z)\sigma_i(w) = -\sigma_i(w)\rho_j^\pm(z), \quad \tau_j^\pm(z)\sigma_i(w) = -\sigma_i(w)\tau_j^\pm(z).$$

The contractions are given by the following Table 1. The entries in the table are the coefficients  $X_{A,B}(z, w)$  in the product  $A(z)B(w) = X_{A,B}(z, w) : A(z)B(w) :$ , where  $A(z)$  is taken from the first column and  $B(w)$  from the first row. A similar convention will be used for other tables.

	$\sigma_0(w)$	$\sigma_1(w)$	$\sigma_2(w)$	$\sigma_3(w)$
$\rho_1^\pm(z)$	$q^{\pm(k_3+1)}z - w$	$q^{\pm(k_2+1)}z - w$	$(q^{\mp(k_1+1)}z - w)^{-1}$	$(q^{\mp 1}z - w)^{-1}$
$\tau_1^\pm(z)$	$(q^{\mp(k_1+1)}z - w)^{-1}$	$(q^{\mp 1}z - w)^{-1}$	$q^{\pm(k_3+1)}z - w$	$q^{\pm(k_2+1)}z - w$

TABLE 1.

Contractions for  $\rho_i^\pm(z)$  and  $\tau_i^\pm(z)$

$$\begin{aligned}
\rho_1^{\epsilon_1}(z)\rho_1^{\epsilon_2}(w) &= z^{\frac{2}{k_1+2}}q^{-\epsilon_1}\frac{(p_1q^{-2}w/z;p_1)_\infty}{(q^2w/z;p_1)_\infty}f_{\epsilon_1\epsilon_2}(w/z) : e^{r_1^{\epsilon_1}(z)+r_1^{\epsilon_2}(w)} :, \\
\tau_1^{\epsilon_1}(z)\tau_1^{\epsilon_2}(w) &= z^{\frac{2}{k_1+2}}q^{-\epsilon_1}\frac{(p_1q^{-2}w/z;p_1)_\infty}{(q^2w/z;p_1)_\infty}f_{\epsilon_1\epsilon_2}(w/z) : e^{t_1^{\epsilon_1}(z)+t_1^{\epsilon_2}(w)} :, \\
\tau_1^{\epsilon_1}(z)\rho_1^{\epsilon_2}(w) &= z^{-\frac{2}{k_1+2}}q^{\epsilon_1}\frac{(q^{-k_2}w/z, p_1q^{k_2+2}w/z;p_1)_\infty}{(q^{-k_2-2}w/z, p_1q^{k_2}w/z;p_1)_\infty}g_{\epsilon_1\epsilon_2}(w/z) : e^{t_1^{\epsilon_1}(z)+r_1^{\epsilon_2}(w)} :, \\
\rho_1^{\epsilon_1}(z)\tau_1^{\epsilon_2}(w) &= z^{-\frac{2}{k_1+2}}q^{\epsilon_1}\frac{(q^{-k_2}w/z, p_1q^{k_2+2}w/z;p_1)_\infty}{(q^{-k_2-2}w/z, p_1q^{k_2}w/z;p_1)_\infty}g_{\epsilon_1\epsilon_2}(w/z) : e^{r_1^{\epsilon_1}(z)+t_1^{\epsilon_2}(w)} :,
\end{aligned}$$

where

$$\begin{aligned}
f_{\pm,\pm}(z) &= 1 - z, \quad f_{\pm,\mp}(z) = 1 - q^{\pm 2}z, \\
g_{\pm,\pm}(z) &= 1, \quad g_{\pm,\mp}(z) = \frac{1 - q^{\mp(k_2+2)}z}{1 - q^{\mp k_2}z}.
\end{aligned}$$

Contractions for  $\rho_i^\pm(z)$  and  $\tau_j^\pm(z)$  ( $i \neq j$ )

Non-trivial contractions are as follows, where double-sign correspondence is implied (notice the upper index in  $\tau_1^\mp$ ).

	$\rho_1^\pm(w)$	$\rho_2^\pm(w)$	$\tau_1^\mp(w)$	$\tau_2^\pm(w)$
$\rho_1^\pm(z)$	*	$q^{\mp 1}\frac{z - q^{\mp k_2}w}{z - q^{\mp(k_2+2)}w}$	*	$q^{\pm 1}\frac{z - q^{\pm k_1}w}{z - q^{\pm(k_1+2)}w}$
$\rho_2^\pm(z)$	$q^{\pm 1}\frac{z - q^{\pm k_2}w}{z - q^{\pm(k_2+2)}w}$	*	$q^{\mp 1}\frac{z - q^{\pm 2}w}{z - w}$	*
$\tau_1^\mp(z)$	*	$q^{\pm 1}\frac{z - q^{\mp 2}w}{z - w}$	*	$q^{\mp 1}\frac{z - q^{\pm(k_3+2)}w}{z - q^{\pm k_3}w}$
$\tau_2^\pm(z)$	$q^{\mp 1}\frac{z - q^{\mp k_1}w}{z - q^{\mp(k_1+2)}w}$	*	$q^{\pm 1}\frac{z - q^{\mp(k_3+2)}w}{z - q^{\mp k_3}w}$	*

TABLE 2.

Asterisks indicate cases with the same subindex, which have been treated before. Contractions not shown in the Table (up to permutation of indices 1, 2, 3) are trivial; for example we have  $\rho_1^\pm(z)\tau_2^\mp(w) =: e^{r_1^\pm(z)+t_2^\mp(w)} :$ .

Contractions for  $\rho_1^\pm(z)$ ,  $\sigma_2(z)$ ,  $\sigma_3(z)$  with generators of  $\mathcal{E}_{2,1}$  We set

$$U_1^\pm(z) =: e^{u_1^\pm(z)} :, \quad V_1^\pm(z) =: e^{v_1^\pm(z)} :, \quad \hat{U}_1^\pm(z) =: e^{\hat{u}_1^\pm(z)} :, \quad \hat{V}_1^\pm(z) =: e^{\hat{v}_1^\pm(z)} :.$$

We have equalities for  $i = 2, 3$

$$\begin{aligned} U^{\varepsilon_1}(z)\rho_1^{\varepsilon_2}(w) &= \rho_1^{\varepsilon_2}(w)U^{\varepsilon_1}(z), & V^{\varepsilon_1}(z)\rho_1^{\varepsilon_2}(w) &= \rho_1^{\varepsilon_2}(w)V^{\varepsilon_1}(z), \\ \hat{U}^{\varepsilon_1}(z)\rho_1^{\varepsilon_2}(w) &= \rho_1^{\varepsilon_2}(w)\hat{U}^{\varepsilon_1}(z), & \hat{V}^{\varepsilon_1}(z)\rho_1^{\varepsilon_2}(w) &= \rho_1^{\varepsilon_2}(w)\hat{V}^{\varepsilon_1}(z), \\ U^\pm(z)\sigma_i(w) &= -\sigma_i(w)U^\pm(z), & V^\pm(z)\sigma_i(w) &= -\sigma_i(w)V^\pm(z), \\ \hat{U}^\pm(z)\sigma_i(w) &= -\sigma_i(w)\hat{U}^\pm(z), & \hat{V}^\pm(z)\sigma_i(w) &= -\sigma_i(w)\hat{V}^\pm(z). \end{aligned}$$

	$\rho_1^\pm(w)$	$\rho_1^\mp(w)$	$\sigma_2(w)$	$\sigma_3(w)$
$U_1^\pm(z)$	$q^{\pm 1} \frac{1 - q^{\mp 2} w/z}{1 - w/z}$	$q^{\pm 1}$	$q^{\mp(k_1+1)} z - w$	$\frac{1}{q^{\pm 1} z - w}$
$V_1^\pm(z)$	$q^{\mp 1} \frac{1 - q^{\mp k_1} w/z}{1 - q^{\mp(k_1+2)} w/z}$	$q^{\mp 1}$	$\frac{1}{q^{\mp 1} z - w}$	$q^{\pm(k_1+1)} z - w$
$\hat{U}_1^\mp(z)$	$q^{\mp 1} \frac{1 - q^{\mp k_1} w/z}{1 - q^{\mp(k_1+2)} w/z}$	$q^{\mp 1}$	$\frac{1}{q^{\mp 1} z - w}$	$q^{\pm(k_1+1)} z - w$
$\hat{V}_1^\mp(z)$	$q^{\pm 1} \frac{1 - q^{\mp 2} w/z}{1 - w/z}$	$q^{\pm 1}$	$q^{\mp(k_1+1)} z - w$	$\frac{1}{q^{\pm 1} z - w}$

TABLE 3.

**Acknowledgments.** The research of BF is supported by the Russian Science Foundation grant project 16-11-10316. MJ is partially supported by JSPS KAKENHI Grant Number JP16K05183. EM is partially supported by a grant from the Simons Foundation #353831.

The authors would like to thank Kyoto University for hospitality during their visits when this work was started.

## REFERENCES

- [ACL] T. Arakawa, T. Creutzig, and A. Linshaw, *W-algebras as coset vertex algebras*, ArXiv:1801.03822, 1–38
- [BKL] V. Bazhanov, G. Kotousov, and S. Lukyanov, *Quantum transfer-matrices for the sausage model*, J. High Energy Phys. (2018), no. 1, 021, front matter+88 pp
- [BL] V. Bazhanov and S. Lukyanov, *Integrable structure of quantum field theory: classical flat connections versus quantum stationary states*, JHEP **1409** (2014), 147, front matter+67 pp
- [BLZ] V. Bazhanov, S. Lukyanov, and A. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz*, Commun. Math. Phys. **177** (1996), no. 2, 381–398
- [Fa] V. Fateev, *The sigma model (dual) representation for a two-parameter family of integrable quantum field theories*, Nucl. Phys. **B473** (1996), 509–538
- [FF] B. Feigin and E. Frenkel, *Representations of affine Kac-Moody algebras and bosonization*, Physics and Mathematics of strings, World Sci. Publishing, Teaneck, NJ (1990), 271–316
- [FF1] B. Feigin and E. Frenkel, *Integrals of Motion and Quantum Groups*, in: Francaviglia M., Greco S. (eds) Integrable Systems and Quantum Groups. Lecture Notes in Mathematics, **1620** (1996), 349–418 Springer, Berlin, Heidelberg
- [FJM] B. Feigin, M. Jimbo, and E. Mukhin, *Integrals of motion from quantum toroidal algebras*, J.Phys.A: Math. Theor., **50** (2017), 464001
- [FJM1] B. Feigin, M. Jimbo, and E. Mukhin, *An evaluation homomorphism for quantum toroidal  $\mathfrak{gl}(n)$  algebras*, ArXiv:1709.01592, 1–12

- [FJM2] B. Feigin, M. Jimbo, and E. Mukhin, *The  $(\mathfrak{gl}_m, \mathfrak{gl}_n)$  duality in the quantum toroidal setting*, ArXiv:1801.08433v1, 1–28
- [FJMM] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Quantum toroidal  $\mathfrak{gl}_1$  and Bethe ansatz*, J. Phys. A **48** (2015), no. 24, 244001, 27 pp
- [FJMM1] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Finite type modules and Bethe ansatz for the quantum toroidal  $\mathfrak{gl}_1$* , Comm. Math. Phys. **356** (2017), no. 1, 285–327
- [FS] B. Feigin and A. Semikhatov, *The  $\widehat{\mathfrak{sl}}(2) \oplus \widehat{\mathfrak{sl}}(2)/\widehat{\mathfrak{sl}}(2)$  coset theory as a Hamiltonian reduction of  $\widehat{D}(2|1; \alpha)$* , Nucl. Phys., **B610** (2001), 489
- [HSTY] I. Heckenberger, F. Spill, A. Torrielli, and H. Yamane, *Drinfeld second realization of the quantum affine superalgebras of  $D^{(1)}(2, 1; x)$  via the Weyl groupoid*, RIMS Kôkyûroku Bessatsu, **B8** (2008), 171–216
- [LZ] S. Lukyanov and A. Zamolodchikov, *Integrable boundary interaction in 3D target space: the “pillow-brane” model*, Nucl. Phys. **B873** (2013), 585–613
- [Mi] K. Miki, *Toroidal and level 0  $U_q \widehat{\mathfrak{sl}}_{n+1}$  actions on  $U_q \widehat{\mathfrak{gl}}_{n+1}$  modules*, J. Math. Phys., **40** (1999), no. 6, 3191–3210
- [Sh] J. Shiraishi, *Free boson representation of  $U_q \widehat{\mathfrak{sl}}_2$* , Phys. Lett., **A171** (1992), 243–248
- [W] M. Wakimoto, *Fock representations of the affine Lie algebra  $A_1^{(1)}$* , Comm. Math. Phys. **104** (1986) 605–609

BF: NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, RUSSIAN FEDERATION, INTERNATIONAL LABORATORY OF REPRESENTATION THEORY AND MATHEMATICAL PHYSICS, RUSSIA, MOSCOW, 101000, MYASNITSKAYA UL., 20 AND LANDAU INSTITUTE FOR THEORETICAL PHYSICS, RUSSIA, CHERNOGOLOVKA, 142432, PR.AKADEMIKA SEMENOVA, 1A  
*E-mail address:* bfeigin@gmail.com

MJ: DEPARTMENT OF MATHEMATICS, RIKKYO UNIVERSITY, TOSHIMA-KU, TOKYO 171-8501, JAPAN  
*E-mail address:* jimbomm@rikkyo.ac.jp

EM: DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY-PURDUE UNIVERSITY-INDIANAPOLIS, 402 N.BLACKFORD ST., LD 270, INDIANAPOLIS, IN 46202, USA  
*E-mail address:* emukhin@iupui.edu